ECE 302: Probability and Applications[1]

Week 2 Topics

- Computing Probability by Counting
- Conditional Probability
- Bayes' Rule

1 Computing Probabilities Using Counting Methods

In many random experiments with finite sample space of size n, *the outcomes are equiprobable.*

- If an event contains k distinct outcomes, then its probability is the ratio of the number of outcomes in the event of interest to the total number of outcomes in the sample space, that is k/n (**Eq. 2.12**).

- To find the probability of an event, we need to count the number outcomes in the event.

- We now develop some useful counting (combinatorial) formulas.

1.1 Ordered Samples

- A multiple-choice test has k questions and question i gives n_i possible answers. What is the total number of ways of answering the entire test?

- Arrange the answers in an ordered k-tuple (entry j in the k-tuple is the answer to question j then the question becomes: **How many distinct ordered k-tuples** $(x_1, \ldots, x_k)$ are possible if x_i is an element from a set with n_i distinct elements?

- Let $k = 2$. If we arrange all possible choices for x_1 and for x_2 along the sides of a table as shown in Fig. 1, we see that there are $n_1 n_2$ distinct ordered pairs. For triplets we could arrange the $n_1 n_2$ possible pairs (x_1, x_2) along the vertical side of the table and the n_3 choices for x_3 along the horizontal side. Clearly, the number of possible triplets is $n_1 n_2 n_3$.

- In general, the **multiplication rule** gives *the number of distinct ordered* k*-tuples* $(x_1, \ldots, x_k)$ *with components* x_i *from a set with* n_i *distinct elements*

$$\text{number of distinct ordered } k\text{-tuples} = n_1 n_2 \ldots n_k. \tag{1}$$

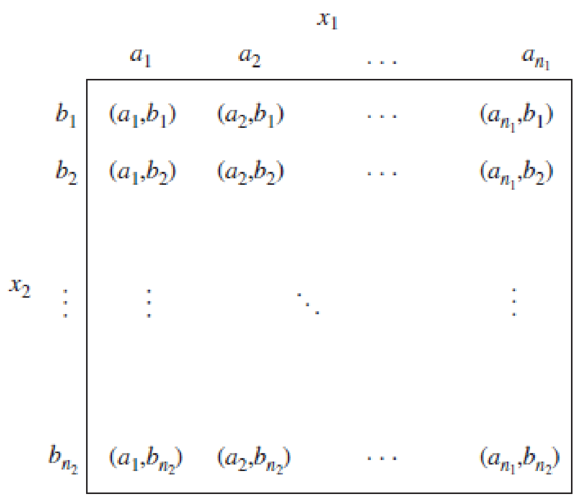

Figure 1: If there are n_1 distinct choices for x_1 and n_2 distinct choices for x_2, then there are $n_1 n_2$ distinct ordered pairs(x_1, x_2).

- Many counting problems can be posed as sampling problems where we select "balls" from "urns" or "objects" from "populations." We will now use Eq. (1) to develop combinatorial formulas for various types of sampling.

1.2 Sampling with Replacement and with Ordering

- We choose k objects from a set A that has n distinct objects, with replacement, that is, after selecting an object and noting its identity in an ordered list, the object is placed back in the set before the next choice is made.

- We will refer to the set A as the "population."

- The experiment produces an ordered k-tuple: $(x_1, \ldots, x_k)$, where $x_i \in A$ and $i = 1, \ldots, k$. Equation (1) with $n_1 = n_2 = \cdots = n_k = n$ implies that

$$\text{number of distinct ordered } k\text{-tuples } = n^k. \tag{2}$$

Example 1

An urn contains five balls numbered 1 to 5. Suppose we select two balls from the urn with replacement. How many possible distinct ordered pairs are possible? What is the probability that the two draws yield the same number?

Equation (2) states that the number of ordered pairs is $5^2 = 25$. Figure 2(a) shows the 25 possible pairs. Five of the 25 outcomes have the two draws yielding the same number; if we suppose that all pairs are equiprobable, then the probability that the two draws yield the same number is $5/25 = .2$.

1.3 Sampling without Replacement and with Ordering

- We choose k objects in succession without replacement from a population A of n distinct objects.
- Clearly, $k \leq n$.
- The number of possible outcomes in the first draw is $n_1 = n$; the number of possible outcomes in the second draw is $n_2 = n-1$, namely all n objects except the one selected in the first draw; and so on up to $n_k = n-(k-1)$ in the final draw. Equation (1) then gives

$$\text{number of distinct ordered } k\text{-tuples } = n(n-1)\ldots(n-k+1). \quad (3)$$

Example 2

An urn contains five balls numbered 1 to 5. Suppose we select two balls in succession without replacement. How many possible distinct ordered pairs are possible? What is the probability that the first ball has a number larger than that of the second ball?

Eq. (3) states the number of ordered pairs is $5(4) = 20$ which are shown in Fig. 2(b). Ten ordered pairs in Fig. 2(b) have the first number larger than the second number; thus the probability of this event is $10/20 = 1/2$.

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5)
\end{array}$$

(a) Ordered pairs for sampling with replacement.

$$\begin{array}{ccccc}
 & (1,2) & (1,3) & (1,4) & (1,5) \\
(2,1) & & (2,3) & (2,4) & (2,5) \\
(3,1) & (3,2) & & (3,4) & (3,5) \\
(4,1) & (4,2) & (4,3) & & (4,5) \\
(5,1) & (5,2) & (5,3) & (5,4) &
\end{array}$$

(b) Ordered pairs for sampling without replacement.

$$\begin{array}{cccc}
(1,2) & (1,3) & (1,4) & (1,5) \\
 & (2,3) & (2,4) & (2,5) \\
 & & (3,4) & (3,5) \\
 & & & (4,5)
\end{array}$$

(c) Pairs for sampling without replacement or ordering.

Figure 2: Enumeration of possible outcomes in various types of sampling of two balls from an urn containing five distinct balls.

Example 3

An urn contains five balls numbered $1, 2, \ldots, 5$. We draw three balls with replacement. What is the probability that all three balls are different?

From Eq. (2) there are $5^3 = 125$ possible outcomes, which we will suppose are equiprobable. The number of these outcomes for which the three draws are different is given by Eq. (3): $5(4)(3) = 60$. Thus the probability that all three balls are different is $60/125 = .48$.

1.4 Permutations of n Distinct Objects

- Consider sampling without replacement with $k = n$. We simply draw objects from an urn containing n distinct objects until the urn is empty.

- The number of possible orderings (arrangements, permutations) of n distinct objects is equal to the number of ordered n-tuples in sampling without replacement with $k = n$. From Eq. (3), we have

$$\text{number of permutations of } n \text{ objects } = n(n-1)\ldots(2)(1) \triangleq n!. \quad (4)$$

- We refer to $n!$ as ***n* factorial**.

- We factorial $n!$ appears in many of the combinatorial formulas.

- For large n, Stirling's formula is very useful:

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n} \quad (5)$$

  where the sign $\sim$ indicates that the ratio of the two sides tends to unity as $n \to \infty$.

Example 4

Find the number of permutations of three distinct objects $\{1, 2, 3\}$. Equation (4) gives $3! = 3(2)(1) = 6$. The six permutations are
123 312 231 132 213 321.

Example 5

Given that 12 airplane crashes occur at random in a year, what is the probability that there is exactly 1 crash each month? The above result shows that this probability is very small. This question is equivalent to: balls are placed at random into 12 cells, where more than 1 ball is allowed to occupy a cell. What is the probability that all cells are occupied?

The placement of each ball into a cell can be viewed as the selection of a cell number between 1 and 12. Equation (2) implies that there are 12^{12} possible placements of the 12 balls in the 12 cells. In order for all cells to be occupied, the first ball selects from any of the 12 cells, the second ball from the remaining 11 cells, and so on. Thus the number of placements that occupy all cells is 12 !. If we suppose that all 12^{12} possible placements are equiprobable, we find that the probability that all cells are occupied is

$$\frac{12!}{12^{12}} = \left(\frac{12}{12}\right)\left(\frac{11}{12}\right)\ldots\left(\frac{1}{12}\right) = 5.37\left(10^{-5}\right)$$

1.5 Sampling without Replacement and without Ordering

- We pick k objects from a set (population) of n distinct objects without replacement and we record the result without regard to order.
- You can imagine putting each selected object into another jar, so that when the k selections are completed we have no record of the order in which they were selected.
- Let C_k^n denote the number of subpopulations of size k from a set of size n.
- We call the specific subset of k selected objects a "subpopulation of size k".
- From Eq. (4), there are $k!$ possible orders in which the k objects in the second jar could have been selected.
- Therefore $C_k^n k!$ must be the total number of distinct ordered samples of k objects, which is given by Eq. (3):

$$C_k^n k! = n(n-1)\dots(n-k+1), \tag{6}$$

- And so the *number of subpopulations of size k from a population of size n, $k \leqslant n$, is*

$$C_k^n = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} \triangleq \binom{n}{k} \tag{7}$$

- The expression $\binom{n}{k}$ is called a **binomial coefficient**.

Example 6

Find the number of ways of selecting two objects from $A = \{1, 2, 3, 4, 5\}$ without regard to order.

Equation (7) gives

$$\binom{5}{2} = \frac{5!}{2!3!} = 10$$

Figure 2(c) gives the 10 pairs.

Example 7

Find the number of distinct permutations of k white balls and $n-k$ black balls.

This problem is equivalent to the following sampling problem: Put n tokens numbered 1 to n in an urn, where each token represents a position in the arrangement of balls; pick a subpopulation of k tokens and put the k white balls in the corresponding positions. Each subpopulation of size k leads to a distinct arrangement (permutation) of k white balls and $n-k$ black balls. Thus the number of distinct permutations of k white balls and $n-k$ black balls is C_k^n.

As a specific example let $n=4$ and $k=2$. The number of subpopulations of size 2 from a set of four distinct objects is

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4(3)}{2(1)} = 6$$

The 6 distinct permutations with 2 whites (zeros) and 2 blacks (ones) are

1100 0110 0011 1001 1010 0101.

Example 8: Quality Control

A batch of 50 items contains 10 defective items. Suppose 10 items are selected at random and tested. What is the probability that exactly 5 of the items tested are defective?

The number of ways of selecting 10 items out of a batch of 50 is the number of subpopulations of size 10 from a set of 50 objects:

$$\binom{50}{10} = \frac{50!}{10!40!}$$

The number of ways of selecting 5 defective and 5 nondefective items from the batch of 50 is the product N_1N_2, where N_1 is the number of ways of selecting the 5 items from the set of 10 defective items, and N_2 is the number of ways of selecting 5 items from the 40 nondefective items. Thus the probability that exactly 5 tested items are defective is

$$\frac{\binom{10}{5}\binom{40}{5}}{\binom{50}{10}} = \frac{10!40!10!40!}{5!5!35!5!50!} = .016$$

- Example 7 shows that sampling without replacement and without ordering is equivalent to partitioning the set of n distinct objects into two sets: B, containing the k items that are picked from the urn, and B^c, containing the $n-k$ left behind.
- Suppose we partition a set of n distinct objects into J subsets $B_1, B_2, \ldots, B_J$, where B_J is assigned k_J elements and $k_1 + k_2 + \cdots + k_J = n$.
- It can be shown that the number of distinct partitions is

$$\frac{n!}{k_1!k_2!\ldots k_J!} \tag{8}$$

  Equation (8) is called the **multinomial coefficient**. The binomial coefficient is the $J = 2$ case of the multinomial coefficient.

Example 9

A six-sided die is tossed 12 times. How many distinct sequences of faces (numbers from the set $\{1, 2, 3, 4, 5, 6\}$) have each number appearing exactly twice? What is the probability of obtaining such a sequence?

The number of distinct sequences in which each face of the die appears exactly twice is the same as the number of partitions of the set $\{1, 2, \ldots, 12\}$ into 6 subsets of size 2, namely

$$\frac{12!}{2!2!2!2!2!2!} = \frac{12!}{2^6} = 7,484,400.$$

From Eq. (2) we have that there are 6^{12} possible outcomes in twelve tosses of a die. If we suppose that all of these have equal probabilities, then the probability of obtaining a sequence in which each face appears exactly twice is

$$\frac{12!/2^6}{6^{12}} = \frac{7,484,400}{2,176,782,336} \simeq 3.4\left(10^{-3}\right).$$

1.6 Sampling with Replacement and without Ordering

- Suppose we pick k objects from a set of n distinct objects with replacement and we record the result without regard to order.

- We can do this by filling out a form which has n columns, one for each distinct object. Each time an object is selected, an "x" is placed in the corresponding column. For example, if we are picking five objects from four distinct objects, one possible form would look like this: xx//x/xx, where the slash symbol ("/") is used to separate the entries for different columns.

- In general, a form has $n-1$ /'s and k x's.

- The number of possible forms is simply the number of possible combinations of $n-1$ 1's (that is, /'s) and k 0's (that is, x's), which is given by the binomial coefficient.

- Therefore, the number of possible ways of picking k objects from a set of n distinct objects with replacement and without ordering is:

$$C_k^{n-1+k} = \binom{n-1+k}{k} = \binom{n-1+k}{n-1} \tag{9}$$

2 Conditional Probability

Frequently we wish to know whether two events, A and B, are related in the sense that knowledge about the occurrence of one, say B, alters the likelihood of occurrence of the other, A.

- To answer this we need to compare the **conditional probability**, $P[A \mid B]$, of event A given that event B has occurred to the event's original probability, $P[A]$.

- The conditional probability is defined by

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} \quad \text{for } P[B] > 0 \tag{10}$$

- If we know that event B has occurred this implies that the outcome is in the set B, that is, the sample space is reduced to B as shown in Fig. 3.

- The event A can now occur if and only if the outcome ζ is in $A \cap B$.

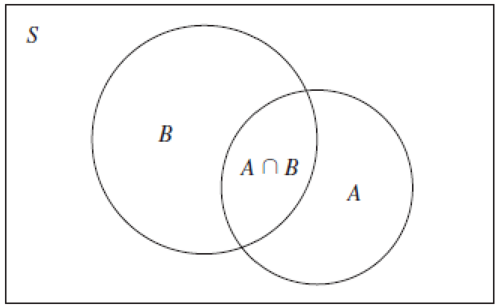

Figure 3: If B is known to have occurred, then A can occur only if $A \cap B$ occurs

- Equation (10) simply renormalizes the probability of events that occur jointly with B. Thus if we let $A = B$, Eq. (10) gives $P[B \mid B] = 1$, as required.
- In terms of relative frequency, then $P[A \mid B]$ is the relative frequency of the event $A \cap B$ in experiments where B occurred.
- If the experiment is performed n times, and if B occurs n_B times, and event $A \cap B, n_{A\cap B}$ times, then the relative frequency of interest is then

$$\frac{n_{A\cap B}}{n_B} = \frac{n_{A\cap B}/n}{n_B/n} \to \frac{P[A \cap B]}{P[B]}.$$

Example 10

A ball is selected from an urn containing two black balls, numbered 1 and 2, and two white balls, numbered 3 and 4. The number and color of the ball is noted, so the sample space is $\{(1, b), (2, b), (3, w), (4, w)\}$. Assuming that the four outcomes are equally likely, find $P[A \mid B]$ and $P[A \mid C]$, where A, B, and C are the following events:

$$\begin{aligned} A &= \{(1, b), (2, b)\}, && \text{"black ball selected,"} \\ B &= \{(2, b), (4, w)\}, && \text{"even-numbered ball selected," and} \\ C &= \{(3, w), (4, w)\}, && \text{"number of ball is greater than 2."} \end{aligned}$$

Since $P[A \cap B] = P[(2, b)]$ and $P[A \cap C] = P[\varnothing] = 0$, Eq. (2.21) gives

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{.25}{.5} = .5 = P[A]$$
$$P[A \mid C] = \frac{P[A \cap C]}{P[C]} = \frac{0}{.5} = 0 \neq P[A].$$

In the first case, knowledge of B did not alter the probability of A. In the second case, knowledge of C implied that A had not occurred.

If we multiply both sides of the definition of $P[A \mid B]$ by $P[B]$ we obtain

$$P[A \cap B] = P[A \mid B]P[B]. \tag{11}$$

Similarly we also have that

$$P[A \cap B] = P[B \mid A]P[A] \tag{12}$$

The next example shows how this equation is useful in finding probabilities in sequential experiments. The example also introduces a tree diagram that facilitates the calculation of probabilities.

Example 11

An urn contains two black balls and three white balls. Two balls are selected at random from the urn without replacement and the sequence of colors is noted. Find the probability that both balls are black.

This experiment consists of a sequence of two subexperiments. We can imagine working our way down the tree shown in Fig. 4 from the topmost node to one of the bottom nodes. We reach node 1 in the tree if the outcome of the first draw is a black ball; then the next subexperiment consists of selecting a ball from an urn containing one black ball and three white balls. On the other hand, if the outcome of the first draw is white, then we reach node 2 in the tree and the second subexperiment consists of selecting a ball from an urn that contains two black balls and two white balls. Thus if we know which node is reached after the first draw, then we can state the probabilities of the outcome in the next subexperiment.

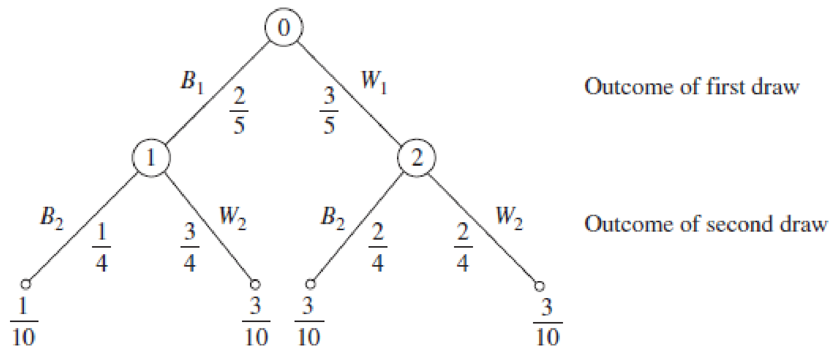

Figure 4: The paths from the top node to a bottom node correspond to the possible outcomes in the drawing of two balls from an urn without replacement. The probability of a path is the product of the probabilities in the associated transitions.

Let B_1 and B_2 be the events that the outcome is a black ball in the first and second draw respectively. From Eq. (12) we have

$$P[B_1 \cap B_2] = P[B_2 \mid B_1] \, P[B_1] .$$

In terms of the tree diagram in Fig. 4, $P[B_1]$ is the probability of reaching node 1 and $P[B_2 \mid B_1]$ is the probability of reaching the leftmost bottom node from node 1. Now $P[B_1] = 2/5$ since the first draw is from an urn containing two black balls and three white balls; $P[B_2 \mid B_1] = 1/4$ since, given B_1, the second draw is from an urn containing one black ball and three white balls. Thus

$$P[B_1 \cap B_2] = \frac{1}{4}\frac{2}{5} = \frac{1}{10}.$$

In general, the probability of any sequence of colors is obtained by multiplying the probabilities corresponding to the node transitions in the tree in Fig. 4.

Example 12

Many communication systems can be modeled in the following way. First, the user inputs a 0 or a 1 into the system, and a corresponding signal is transmitted. Second, the receiver makes a decision about what was the input to the system, based on the signal it received. Suppose that the user sends 0 s with probability $1-p$ and 1 s with probability p, and suppose that the receiver makes random decision errors with probability ε. For $i = 0, 1$ let A_i be the event "input was i," and let B_i be the event "receiver decision was i." Find the probabilities $P[A_i \cap B_j]$ for $i = 0, 1$ and $j = 0, 1$.

The tree diagram for this experiment is shown in Fig. 5. We then readily obtain the desired probabilities

$$
\begin{aligned}
&P[A_0 \cap B_0] = (1-p)(1-\varepsilon) \\
&P[A_0 \cap B_1] = (1-p)\varepsilon \\
&P[A_1 \cap B_0] = p\varepsilon, \text{ and} \\
&P[A_1 \cap B_1] = p(1-\varepsilon)
\end{aligned}
$$

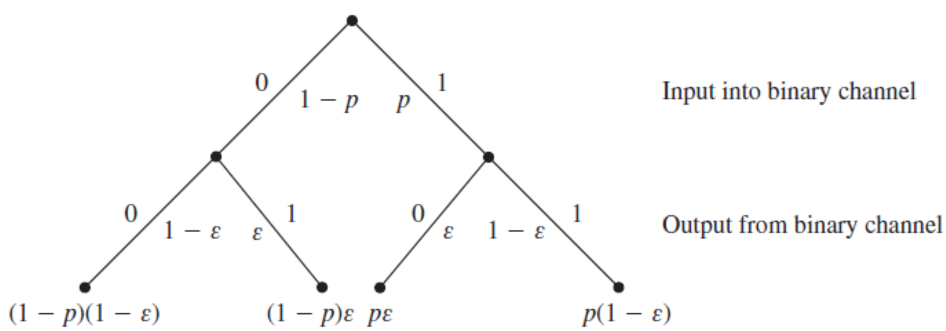

Figure 5: Probabilities of input-output pairs in a binary transmission system

2.1 Theorem on Total Probability

- Let $B_1, B_2, \ldots, B_n$ be mutually exclusive events whose union equals the sample space S as shown in Fig. 6.
- We refer to these sets as a **partition** of S.
- Any event A can be represented as the union of mutually exclusive events in the following way:

$$
\begin{aligned}
A &= A \cap S = A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) \\
&= (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n).
\end{aligned}
$$

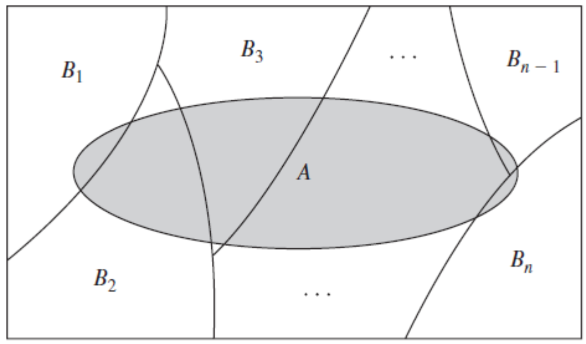

Figure 6: A partition of S into n disjoint sets

- See Fig. 6. By Corollary 4, the probability of A is
  $P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_n]$
- By applying Eq. (11) to each of the terms on the right-hand side, we obtain the **theorem on total probability**:

$$P[A] = P[A \mid B_1] P[B_1] + P[A \mid B_2] P[B_2] + \cdots + P[A \mid B_n] P[B_n] \tag{13}$$

- This result is particularly useful when the experiments can be viewed as consisting of a sequence of two subexperiments as shown in the tree diagram in Fig. 4.

Example 13

In the experiment discussed in Example 11, find the probability of the event W_2 that the second ball is white.

The events $B_1 = \{(b,b),(b,w)\}$ and $W_1 = \{(w,b),(w,w)\}$ form a partition of the sample space, so applying Eq. (13) we have

$$\begin{aligned} P[W_2] &= P[W_2 \mid B_1] P[B_1] + P[W_2 \mid W_1] P[W_1] \\ &= \frac{3}{4}\frac{2}{5} + \frac{1}{2}\frac{3}{5} = \frac{3}{5} \end{aligned}$$

It is interesting to note that this is the same as the probability of selecting a white ball in the first draw. The result makes sense because we are computing the probability of a white ball in the second draw under the assumption that we have no knowledge of the outcome of the first draw.

Example 14

A manufacturing process produces a mix of "good" memory chips and "bad" memory chips. The lifetime of good chips follow the exponential law introduced in Example 8 in Week 1 Notes, with a rate of failure α. The lifetime of bad chips also follows the exponential law but the rate of failure is 1000α. Suppose that the fraction of good chips is $1-p$ and of bad chips, p. Find the probability that a randomly selected chip is still functioning after t seconds.

Let C be the event, "chip still functioning after t seconds," and let G be the event "the chip is good" and B be the event "the chip is bad." By the theorem of total probability we have

$$\begin{aligned} P[C] &= P[C \mid G]P[G] + P[C \mid B]P[B] \\ &= P[C \mid G](1-p) + P[C \mid B]p \\ &= (1-p)e^{-\alpha t} + pe^{-1000\alpha t} \end{aligned}$$

where we used the fact that $P[C \mid G] = e^{-\alpha t}$ and $P[C \mid B] = e^{-1000\alpha t}$.

2.2 Bayes' Rule

- Let $B_1, B_2, \ldots, B_n$ be a partition of a sample space S.
- *Suppose that event A occurs, what is the probability of event B_j ?*
- By the definition of conditional probability we have

$$P\left[B_j \mid A\right] = \frac{P\left[A \cap B_j\right]}{P[A]} = \frac{P\left[A \mid B_j\right] P\left[B_j\right]}{\sum_{k=1}^{n} P\left[A \mid B_k\right] P\left[B_k\right]} \tag{14}$$

- where we used the theorem on total probability to replace $P[A]$.
- Equation (14) is called **Bayes' rule**.
- *Bayes' Rule is often applied when are given partial information about the outcome of an experiment.*
- We some experiment where the events of interest form a partition.
- The "a priori probabilities" of these events, $P\left[B_j\right]$, are the probabilities of the events before the experiment is performed.

- Suppose that the experiment is performed, and we are informed that event A occurred; the "a posteriori probabilities" are the probabilities of the events in the partition, $P[B_j \mid A]$, given this additional information.

The following two examples illustrate this situation.

Example 15: Binary Communication System

In the binary communication system in Example 12, find which input is more probable given that the receiver has output a 1. Assume that, a priori, the input is equally likely 0 or 1.

Let A_k be the event that the input was $k, k = 0, 1$, then A_0 and A_1 are a partition of the sample space of input-output pairs. Let B_1 be the event receiver output was a 1. The probability of B_1 is

$$\begin{aligned} P[B_1] &= P[B_1 \mid A_0]\,P[A_0] + P[B_1 \mid A_1]\,P[A_1] \\ &= \varepsilon\left(\frac{1}{2}\right) + (1-\varepsilon)\left(\frac{1}{2}\right) = \frac{1}{2} \end{aligned}$$

Applying Bayes' rule, we obtain the a posteriori probabilities

$$\begin{aligned} P[A_0 \mid B_1] &= \frac{P[B_1 \mid A_0]\,P[A_0]}{P[B_1]} = \frac{\varepsilon/2}{1/2} = \varepsilon \\ P[A_1 \mid B_1] &= \frac{P[B_1 \mid A_1]\,P[A_1]}{P[B_1]} = \frac{(1-\varepsilon)/2}{1/2} = (1-\varepsilon) \end{aligned}$$

Thus, if ε is less than 1/2, then input 1 is more likely than input 0 when a 1 is observed at the output of the channel.

Example 16: Quality Control

Consider the memory chips discussed in Example 14. Recall that a fraction p of the chips are bad and tend to fail much more quickly than good chips. Suppose that in order to "weed out" the bad chips, every chip is tested for t seconds prior to leaving the factory. The chips that fail are discarded and the remaining chips are sent out to customers. Find the value of t for which 99% of the chips sent out to customers are good.

Let C be the event "chip still functioning after t seconds," and let G be the event "chip is good," and B the event "chip is bad." The problem requires that we find the value of t for which

$$P[G \mid C] = .99$$

We find $P[G \mid C]$ by applying Bayes' rule:

$$\begin{aligned}
P[G \mid C] &= \frac{P[C \mid G]P[G]}{P[C \mid G]P[G] + P[C \mid B]P[B]} \\
&= \frac{(1-p)e^{-\alpha t}}{(1-p)e^{-\alpha t} + pe^{-\alpha 1000t}} \\
&= \frac{1}{1 + \frac{pe^{-\alpha 1000t}}{(1-p)e^{-\alpha t}}} = .99.
\end{aligned}$$

The above equation can then be solved for t :

$$t = \frac{1}{999\alpha} \ln\left(\frac{99p}{1-p}\right).$$

For example if $1/\alpha = 20,000$ hours and $p = .10$, then $t = 48$ hours.