ECE 302: Probability and Applications[1]

Week 8 Topics

- Joint pdf of Two Continuous Random Variables
- Independence of Two Random Variables
- Joint Moments and Expected Value of a Function of Two Random Variables

1 The Joint pdf of Two Continuous Random Variables

The joint cdf allows us to compute the probability of events that correspond to "rectangular" shapes in the plane. To compute the probability of events corresponding to regions other than rectangles, we note that any reasonable shape (i.e., disk, polygon, or half-plane) can be approximated by the union of disjoint infinitesimal rectangles, $B_{j,k}$. For example, Fig. 1 shows how the events $A = \{X + Y \leq 1\}$ and $B = \{X^2 + X^2 \leq 1\}$ are approximated by rectangles of infinitesimal width. The probability of such events can therefore be approximated by the sum of the probabilities of infinitesimal rectangles, and if the cdf is sufficiently smooth, the probability of each rectangle can be expressed in terms of a density function:

$$P[B] \approx \sum_j \sum_k P\left[B_{j,k}\right] = \sum_{(x_j, y_k) \in B} f_{X,Y}\left(x_j, y_k\right) \Delta x \Delta y$$

As Δx and Δy approach zero, the above equation becomes an integral of a probability density function over the region B.

We say that the random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ **are jointly continuous** if the probabilities of events involving (X, Y) can be expressed as an integral of a probability density function. In other words, there is a nonnegative function $f_{X,Y}(x, y)$, called the **joint probability density function**, that is defined on the real plane such that for every event B, a subset of the plane,

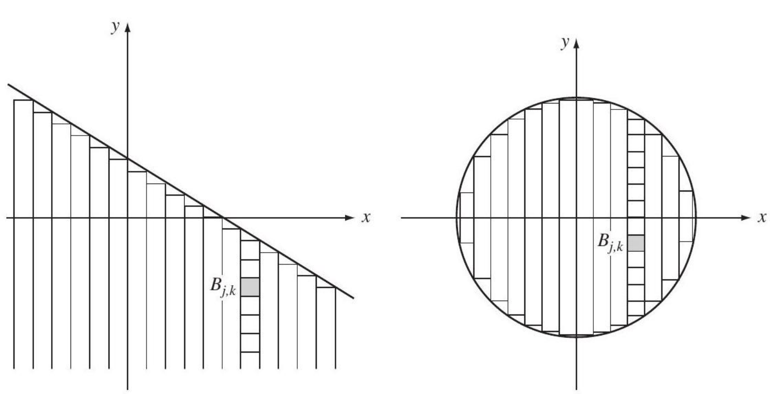

Figure 1: Some two-dimensional non-product form events.

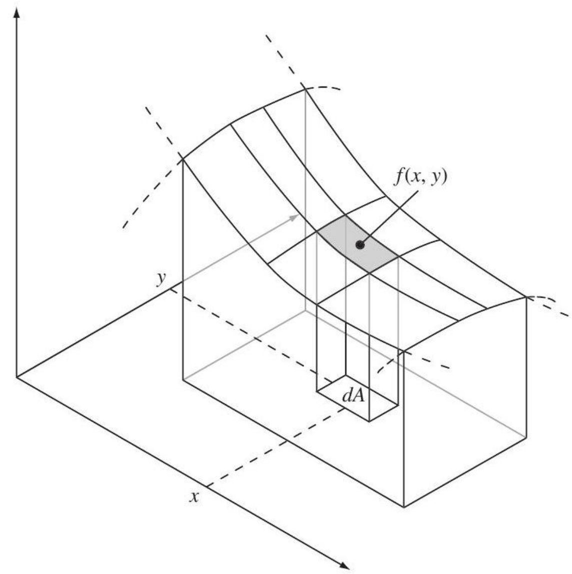

Figure 2: The probability of A is the integral of $f_{X,Y}(x, y)$ over the region defined by A.

$$P[\mathbf{X} \text{ in } B] = \int_B \int f_{X,Y}(x', y')\, dx' dy' \tag{1}$$

as shown in Fig. 2. Note the similarity to the double summations used to find the probability of events for discrete random variables. When B is the entire plane, the integral must equal one:

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x', y')\, dx' dy' \tag{2}$$

Equations (1) and (2) again suggest that the probability "mass" of an event is found by integrating the density of probability mass over the region corresponding to the event.

The joint cdf can be obtained in terms of the joint pdf of jointly continuous random variables by integrating over the semi-infinite rectangle defined by (x, y) :

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x', y')\, dx' dy' \tag{3}$$

It then follows that if X and Y are jointly continuous random variables, then the pdf can be obtained from the cdf by differentiation:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \tag{4}$$

Note that if X and Y are not jointly continuous, then it is possible that the above partial derivative does not exist. In particular, if the $F_{X,Y}(x,y)$ is discontinuous or if its partial derivatives are discontinuous, then the joint pdf as defined by Eq. (4) will not exist at that point.

The probability of a rectangular region is obtained by letting $B = \{(x,y)\colon a_1 < x \leq b_1 \text{ and } a_2 < y \leq b_2\}$ in Eq.(1):

$$P\left[a_1 < X \leq b_1, a_2 < Y \leq b_2\right] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f_{X,Y}\left(x', y'\right) dx' dy' \tag{5}$$

It then follows that the probability of an infinitesimal rectangle is the product of the pdf and the area of the rectangle:

$$\begin{aligned} P[x < X \leq x + dx, y < Y \leq y + dy] &= \int_{x}^{x+dx} \int_{y}^{y+dy} f_{X,Y}\left(x', y'\right) dx' dy' \\ &\simeq f_{X,Y}(x,y) dx dy \end{aligned} \tag{6}$$

The **marginal pdf's** $f_X(x)$ and $f_Y(y)$ are obtained by taking the derivative of the corresponding marginal cdf's, $F_X(x) = F_{X,Y}(x,\infty)$ and $F_Y(y) = F_{X,Y}(\infty,y)$. Thus

$$\begin{aligned} f_X(x) &= \frac{d}{dx} \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{X,Y}\left(x', y'\right) dy' \right\} dx' \\ &= \int_{-\infty}^{\infty} f_{X,Y}\left(x, y'\right) dy' \end{aligned} \tag{7}$$

Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}\left(x', y\right) dx' \tag{8}$$

Thus the marginal pdf's are obtained by integrating out the variables that are not of interest.

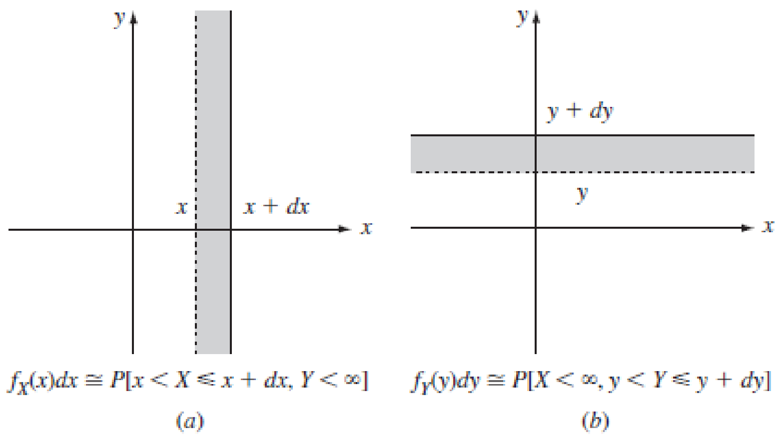

Figure 3: Interpretation of marginal pdf's.

Note that $f_X(x)dx \simeq P[x < X \leq x + dx, Y < \infty]$ is the probability of the infinitesimal strip shown in Fig. 3(a). This reminds us of the interpretation of the marginal pmf's as the probabilities of columns and rows in the case of discrete random variables. As in the case of pmf's, we note that, in general, the joint pdf cannot be obtained from the marginal pdf's.

Example 1: Jointly Uniform Random Variables

A randomly selected point (X, Y) in the unit square has the uniform joint pdf given by

$$f_{X,Y}(x, y) = \begin{cases} 1 & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

A scattergram for this random pair points evently scattered in the unit square. Find the joint cdf of X and Y.

The cdf is found by evaluating Eq. (3). You must be careful with the limits of the integral: The limits should define the region consisting of the intersection of the semi-infinite rectangle defined by (x, y) and the region where the pdf is nonzero. There are five cases in this problem, corresponding to the five regions shown in Fig. 4.

1. If $x < 0$ or $y < 0$, the pdf is zero and Eq. (4) implies

$$F_{X,Y}(x, y) = 0$$

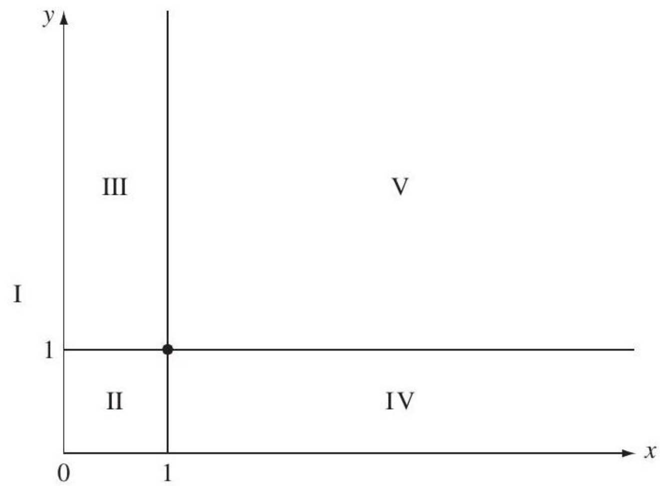

Figure 4: Regions that need to be considered separately in computing cdf in Example 1.

2. If (x, y) is inside the unit interval,

$$F_{X,Y}(x,y) = \int_0^x \int_0^y 1 dx' dy' = xy$$

3. If $0 \leq x \leq 1$ and $y > 1$,

$$F_{X,Y}(x,y) = \int_0^x \int_0^1 1 dx' dy' = x$$

4. Similarly, if $x > 1$ and $0 \leq y \leq 1$,

$$F_{X,Y}(x,y) = y$$

5. Finally, if $x > 1$ and $y > 1$,

$$F_{X,Y}(x,y) = \int_0^1 \int_0^1 1 dx' dy' = 1$$

We see that this is the joint cdf of Example 18 in Week 7.

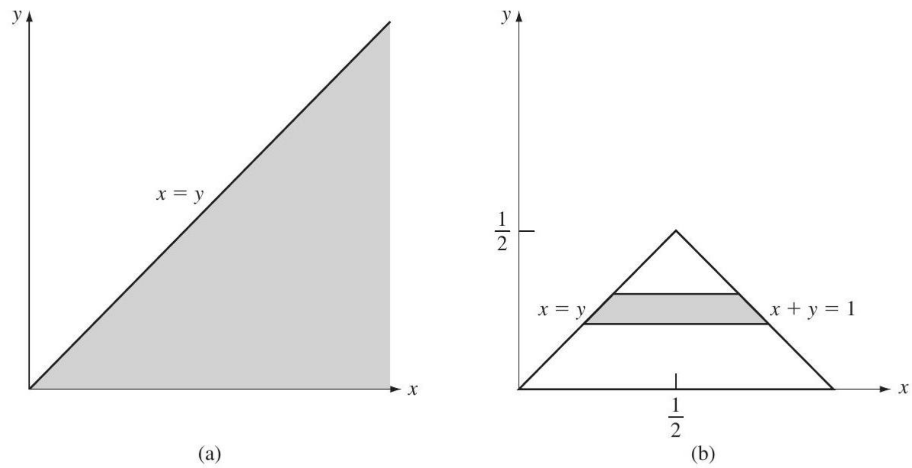

Figure 5: The random variables X and Y in Examples 2 and 3 have a pdf that is nonzero only in the shaded region shown in part (a).

Example 2

Find the normalization constant c and the marginal pdf's for the following joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

The pdf is nonzero in the shaded region shown in Fig. 5(a). The constant c is found from the normalization condition specified by Eq. (2):

$$1 = \int_0^\infty \int_0^x ce^{-x}e^{-y}dydx = \int_0^\infty ce^{-x}\left(1 - e^{-x}\right)dx = \frac{c}{2}$$

Therefore $c = 2$. The marginal pdf's are found by evaluating Eqs. (7) and (8):

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)dy = \int_0^x 2e^{-x}e^{-y}dy = 2e^{-x}\left(1 - e^{-x}\right) \quad 0 \le x < \infty$$

and

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)dx = \int_y^\infty 2e^{-x}e^{-y}dx = 2e^{-2y} \quad 0 \le y < \infty$$

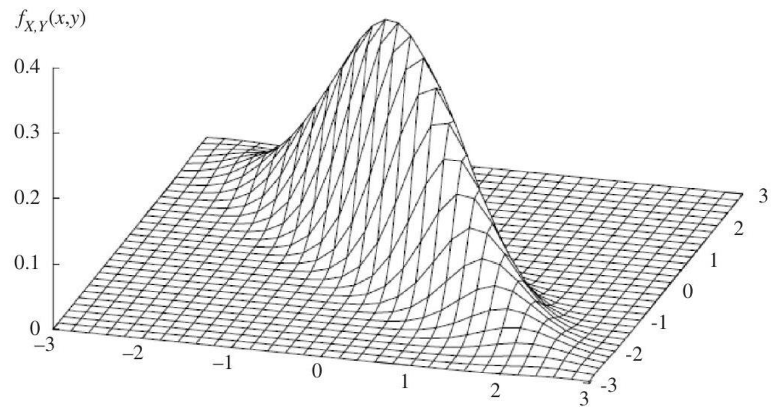

Figure 6: Joint pdf of two jointly Gaussian random variables.

You should fill in the steps in the evaluation of the integrals as well as verify that the marginal pdf's integrate to 1.

Example 3

Find $P[X+Y \leq 1]$ in Example 2.

Figure 5(b) shows the triangular area where the event $\{X+Y \leq 1\}$ intersects with the region where the pdf is nonzero. We obtain the probability of the event by "adding" (actually integrating) infinitesimal rectangles of width dy as indicated in the figure:

$$P[X+Y \leq 1] = \int_0^{.5} \int_y^{1-y} 2e^{-x}e^{-y} dx dy = \int_0^{.5} 2e^{-y} \left[e^{-y} - e^{-(1-y)}\right] dy$$
$$= 1 - 2e^{-1}$$

Example 4: Jointly Gaussian Random Variables

The joint pdf of X and Y, shown in Fig. 6, is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left(x^2 - 2\rho xy + y^2\right)/2\left(1-\rho^2\right)} \qquad -\infty < x, y < \infty \qquad (9)$$

We say that X and Y are jointly Gaussian[2]. Find the marginal pdf's.

[2] This is an important special case of jointly Gaussian random variables. The general case is discussed later.

The marginal pdf of X is found by integrating $f_{X,Y}(x,y)$ over y :

$$f_X(x) = \frac{e^{-x^2/2\left(1-\rho^2\right)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\left(y^2-2\rho xy\right)/2\left(1-\rho^2\right)} dy$$

We complete the square of the argument of the exponent by adding and subtracting $\rho^2 x^2$, that is, $y^2 - 2\rho xy + \rho^2 x^2 - \rho^2 x^2 = (y - \rho x)^2 - \rho^2 x^2$. Therefore

$$\begin{aligned} f_X(x) &= \frac{e^{-x^2/2\left(1-\rho^2\right)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\left[(y-\rho x)^2-\rho^2x^2\right]/2\left(1-\rho^2\right)} dy \\ &= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-(y-\rho x)^2/2\left(1-\rho^2\right)}}{\sqrt{2\pi\left(1-\rho^2\right)}} dy \\ &= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \end{aligned}$$

where we have noted that the last integral equals one since its integrand is a Gaussian pdf with mean ρx and variance $1 - \rho^2$. The marginal pdf of X is therefore a one-dimensional Gaussian pdf with mean 0 and variance 1 . From the symmetry of $f_{X,Y}(x,y)$ in x and y, we conclude that the marginal pdf of Y is also a one-dimensional Gaussian pdf with zero mean and unit variance.

2 Independence of Two Random Variables

$\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent random variables if any event A_1 defined in terms of X is independent of any event A_2 defined in terms of Y; that is,

$$P\left[X \text{ in } A_1, Y \text{ in } A_2\right] = P\left[X \text{ in } A_1\right] P\left[Y \text{ in } A_2\right] \tag{10}$$

In this section we present a simple set of conditions for determining when X and Y are independent.

Suppose that X and Y are a pair of discrete random variables, and suppose we are interested in the probability of the event $A = A_1 \cap A_2$, where A_1 involves only X and A_2 involves only Y. In particular, if X and Y are independent, then A_1 and A_2 are independent events. If we let $A_1 = \{X = x_j\}$ and $A_2 = \{Y = y_k\}$, then the independence of X and Y implies that

$$\begin{aligned} p_{X,Y}(x_j, y_k) &= P[X = x_j, Y = y_k] \\ &= P[X = x_j]\, P[Y = y_k] \\ &= p_X(x_j)\, p_Y(y_k) \qquad \text{for all } x_j \text{ and } y_k \end{aligned} \tag{11}$$

Therefore, *if X and Y are independent discrete random variables, then the joint pmf is equal to the product of the marginal pmf's.*

Now suppose that we don't know if X and Y are independent, but we do know that the pmf satisfies Eq. (11). Let $A = A_1 \cap A_2$ be a product-form event as above, then

$$\begin{aligned} P[A] &= \sum_{x_j \text{ in } A_1} \sum_{y_k \text{ in } A_2} p_{X,Y}(x_j, y_k) \\ &= \sum_{x_j \text{ in } A_1} \sum_{y_k \text{ in } A_2} p_X(x_j) p_Y(y_k) \\ &= \sum_{x_j \text{ in } A_1} p_X(x_j) \sum_{y_k \text{ in } A_2} p_Y(y_k) \\ &= P[A_1]\, P[A_2], \end{aligned} \tag{12}$$

which implies that A_1 and A_2 are independent events. Therefore, *if the joint pmf of X and Y equals the product of the marginal pmf's, then X and Y are independent.* We have just proved that " X and Y are independent" is equivalent to the statement "the joint pmf is equal to the product of the marginal pmf's." In mathematical language, we say, the "discrete random variables X and Y are independent if and only if the joint pmf is equal to the product of the marginal pmf's for all x_j, y_k."

Example 5

Is the pmf in Example 13 in Week 7 consistent with an experiment that consists of the independent tosses of two fair dice?

The probability of each face in a toss of a fair die is $1/6$. If two fair dice are tossed and if the tosses are independent, then the probability of any pair of faces, say j and k, is:

$$P[X = j, Y = k] = P[X = j]P[Y = k] = \frac{1}{36}$$

Thus all possible pairs of outcomes should be equiprobable. This is not the case for the joint pmf given in Example 5.6. Therefore the tosses in Example 5.6 are not independent.

Example 6

Are Q and R in Example 16 of Week 7 independent? From that example we have

$$\begin{aligned} P[Q = q]P[R = r] &= \left(1 - p^M\right)\left(p^M\right)^q \frac{(1-p)}{1-p^M} p^r \\ &= (1-p)p^{Mq+r} \\ &= P[Q = q, R = r] \quad \text{for all } q = 0, 1, \ldots \\ &\qquad\qquad\qquad\qquad\quad r = 0, \ldots, M-1 \end{aligned}$$

Therefore Q and R are independent.

In general, it can be shown that the random variables X and Y are independent if and only if their joint cdf is equal to the product of its marginal cdf's:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) \quad \text{for all } x \text{ and } y \tag{13}$$

Similarly, if X and Y are jointly continuous, then X and Y are independent if and only if their joint pdf is equal to the product of the marginal pdf's:

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) \quad \text{for all } x \text{ and } y \tag{14}$$

Equation (14) is obtained from Eq. (13) by differentiation. Conversely, Eq. (13) is obtained from Eq. (14) by integration.

Example 7

Are the random variables X and Y in Example 2 independent?

Note that $f_X(x)$ and $f_Y(y)$ are nonzero for all $x > 0$ and all $y > 0$. Hence $f_X(x)f_Y(y)$ is nonzero in the entire positive quadrant. However $f_{X,Y}(x, y)$ is nonzero only in the region $y < x$ inside the positive quadrant. Hence Eq. (14) does not hold for all x, y and the random variables are not independent. You should note that in this example the joint pdf appears to factor, but nevertheless it is not the product of the marginal pdf's.

Example 8

Are the random variables X and Y in Example 4 independent? The product of the marginal pdf's of X and Y in Example 4 is

$$f_X(x)f_Y(y) = \frac{1}{2\pi}e^{-\left(x^2+y^2\right)/2} \qquad -\infty < x, y < \infty$$

By comparing to Eq. (9) we see that the product of the marginals is equal to the joint pdf if and only if $\rho = 0$. Therefore the jointly Gaussian random variables X and Y are independent if and only if $\rho = 0$. We see in a later section that ρ is the correlation coefficient between X and Y.

Example 9

Are the random variables X and Y independent in Example 19 in week 7? If we multiply the marginal cdf's found in the Example we find

$$F_X(x)F_Y(y) = \left(1 - e^{-\alpha x}\right)\left(1 - e^{-\beta y}\right) = F_{X,Y}(x, y) \qquad \text{for all } x \text{ and } y$$

Therefore Eq. (13) is satisfied so X and Y are independent.

If X and Y are independent random variables, then the random variables defined by any pair of functions $g(X)$ and $h(Y)$ are also independent. To show this, consider the one-dimensional events A and B. Let A' be the set of all values of x such that if x is in A' then $g(x)$ is in A, and let B' be the set of all values of y such that if y is in B' then $h(y)$ is in B. (Note that A' and B' are equivalent events of A and B.) Then

$$\begin{aligned} P[g(X) \text{ in } A, h(Y) \text{ in } B] &= P\left[X \text{ in } A', Y \text{ in } B'\right] \\ &= P\left[X \text{ in } A'\right]P\left[Y \text{ in } B'\right] \\ &= P[g(X) \text{ in } A]P[h(Y) \text{ in } B]. \end{aligned} \tag{15}$$

The first and third equalities follow from the fact that A and A' and B and B' are equivalent events. The second equality follows from the independence of X and Y. Thus $g(X)$ and $h(Y)$ are independent random variables.

3 Joint Moments and Expected Values of a Function of Two Random Variables

The expected value of X identifies the center of mass of the distribution of X. The variance, which is defined as the expected value of $(X-m)^2$, provides a measure of the spread of the distribution about the mean. In the case of two random variables we are interested in how X and Y vary together. The scattergrams in Figure 4 of Week 7 show a variety of possible joint behaviors. Here we are interested in whether the variation of X and Y are correlated. For example, if X increases does Y tend to increase or to decrease? The joint moments of X and Y, which are defined as expected values of functions of X and Y, provide this information.

3.1 Expected Value of a Function of Two Random Variables

The problem of finding the expected value of a function of two or more random variables is similar to that of finding the expected value of a function of a single random variable. It can be shown that the expected value of $Z = g(X, Y)$ can be found using the following expressions:

$$E[Z] = \begin{cases} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy & X, Y \text{ jointly continuous} \\ \sum_i \sum_n g\left(x_i, y_n\right) p_{X,Y}\left(x_i, y_n\right) & X, Y \text{ discrete.} \end{cases}$$

Example 10: Sum of Random Variables
Let $Z = X + Y$. Find $E[Z]$.

$$\begin{aligned} E[Z] &= E[X+Y] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(x' + y'\right) f_{X,Y}\left(x', y'\right) dx' dy' \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x' f_{X,Y}\left(x', y'\right) dy' dx' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y' f_{X,Y}\left(x', y'\right) dx' dy' \\ &= \int_{-\infty}^{\infty} x' f_X\left(x'\right) dx' + \int_{-\infty}^{\infty} y' f_Y\left(y'\right) dy' = E[X] + E[Y] \end{aligned} \tag{16}$$

Thus the expected value of the sum of two random variables is equal to the sum of the individual expected values. Note that X and Y need not be independent.

The result in Example 10 and a simple induction argument show that *the expected value of a sum of n random variables is equal to the sum of the expected values*:

$$E\left[X_1 + X_2 + \cdots + X_n\right] = E\left[X_1\right] + \cdots + E\left[X_n\right] \tag{17}$$

Note that the random variables do not have to be independent.

Example 11: Product of Functions of Independent Random Variables

Suppose that X and Y are independent random variables, and let $g(X,Y) = g_1(X)g_2(Y)$. Find $E[g(X,Y)] = E\left[g_1(X)g_2(Y)\right]$.

$$\begin{aligned}
E\left[g_1(X)g_2(Y)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1\left(x'\right) g_2\left(y'\right) f_X\left(x'\right) f_Y\left(y'\right) dx'dy' \\
&= \left\{\int_{-\infty}^{\infty} g_1\left(x'\right) f_X\left(x'\right) dx'\right\}\left\{\int_{-\infty}^{\infty} g_2\left(y'\right) f_Y\left(y'\right) dy'\right\} \\
&= E\left[g_1(X)\right] E\left[g_2(Y)\right].
\end{aligned}$$

3.2 Joint Moments, Correlation, and Covariance

The joint moments of two random variables X and Y summarize information about their joint behavior. The jkth **joint moment of *X* and *Y*** is defined by

$$E\left[X^jY^k\right] = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{X,Y}(x,y)dxdy & X,Y \text{ jointly continuous} \\ \sum_i \sum_n x_i^j y_n^k p_{X,Y}\left(x_i, y_n\right) & X,Y \text{ discrete} \end{cases}$$

If $j = 0$, we obtain the moments of Y, and if $k = 0$, we obtain the moments of X.

It is customary to call the $j = 1$ $k = 1$ moment, $E[XY]$, the **correlation of *X* and *Y***. If $E[XY] = 0$, then we say that ***X* and *Y* are orthogonal**.

The jkth **central moment of X and Y** is defined as the joint moment of the centered random variables, $X - E[X]$ and $Y - E[Y]$:

$$E\left[(X - E[X])^j (Y - E[Y])^k\right]$$

Note that $j = 2$ $k = 0$ gives VAR(X) and $j = 0$ $k = 2$ gives VAR(Y).

The **covariance of X and Y** is defined as the $j = k = 1$ central moment:

$$\text{COV}(X, Y) = E[(X - E[X])(Y - E[Y])] \tag{18}$$

The following form for COV(X, Y) is sometimes more convenient to work with:

$$\begin{aligned}\text{COV}(X, Y) &= E[XY - XE[Y] - YE[X] + E[X]E[Y]] \\ &= E[XY] - 2E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned} \tag{19}$$

Note that COV$(X, Y) = E[XY]$ if either of the random variables has mean zero.

Example 12: Covariance of Independent Random Variables

Let X and Y be independent random variables. Find their covariance.

$$\begin{aligned}\text{COV}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[X - E[X]]E[Y - E[Y]] \\ &= 0\end{aligned}$$

where the second equality follows from the fact that X and Y are independent, and the third equality follows from $E[X - E[X]] = E[X] - E[X] = 0$. Therefore *pairs of independent random variables have covariance zero.*

Let's see how the covariance measures how X and Y vary jointly. The covariance measures the average of the product of the deviation of X from $m_X = E[X]$ and the deviation of Y from $m_Y = E[Y]$. If a positive value of $(X - m_X)$ tends to be accompanied by a positive values of $(Y - m_Y)$, and negative $(X - m_X)$ tend to be accompanied by negative $(Y - m_Y)$; then $(X - m_X)(Y - m_Y)$ will tend to be a positive value, and its expected value, $\text{COV}(X, Y)$, will be positive. This is the case for the scattergram in Fig. 4(d) in Week 7 where the observed points tend to cluster along a line of positive slope. On the other hand, if $(X - m_X)$ and $(Y - m_Y)$ tend to have opposite signs, then $\text{COV}(X, Y)$ will be negative. A scattergram for this case would have observation points cluster along a line of negative slope. Finally if $(X - m_X)$ and $(Y - m_Y)$ sometimes have the same sign and sometimes have opposite signs, then $\text{COV}(X, Y)$ will be close to zero. The three scattergrams in Figs.4 (a), (b), and (c) in Week 7 fall into this category.

Multiplying either X or Y by a large number will increase the covariance, so we need to normalize the covariance to measure the correlation in an absolute scale. The **correlation coefficient of $\boldsymbol{X}$ and $\boldsymbol{Y}$** is defined by

$$\rho_{X,Y} = \frac{\text{COV}(X, Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} \tag{20}$$

where $\sigma_X = \sqrt{\text{VAR}(X)}$ and $\sigma_Y = \sqrt{\text{VAR}(Y)}$ are the standard deviations of X and Y, respectively.

The correlation coefficient is a number that is at most 1 in magnitude:

$$-1 \le \rho_{X,Y} \le 1 \tag{21}$$

To show Eq. (21), we begin with an inequality that results from the fact that the expected value of the square of a random variable is nonnegative:

$$\begin{aligned} 0 &\le E\left\{\left(\frac{X - E[X]}{\sigma_X} \pm \frac{Y - E[Y]}{\sigma_Y}\right)^2\right\} \\ &= 1 \pm 2\rho_{X,Y} + 1 \\ &= 2\left(1 \pm \rho_{X,Y}\right) \end{aligned}$$

The last equation implies Eq. (21).

The extreme values of $\rho_{X,Y}$ are achieved when X and Y are related linearly, $Y = aX + b; \rho_{X,Y} = 1$ if $a > 0$ and $\rho_{X,Y} = -1$ if $a < 0$. Later we will show that $\rho_{X,Y}$ can be viewed as a statistical measure of the extent to which Y can be predicted by a linear function of X.

X and Y are said to be **uncorrelated** if $\rho_{X,Y} = 0$. If X and Y are independent, then $\text{COV}(X, Y) = 0$, so $\rho_{X,Y} = 0$. Thus *if X and Y are independent, then X and Y are uncorrelated*. In Example 8, we saw that *if X and Y are jointly Gaussian and $\rho_{X,Y} = 0$, then X and Y are independent random variables.* Example 13 below shows that this is not always true for non-Gaussian random variables: It is possible for X and Y to be uncorrelated but not independent.

Example 13: Uncorrelated but Dependent Random Variables

Let Θ be uniformly distributed in the interval $(0, 2\pi)$. Let

$$X = \cos\Theta \quad \text{and} \quad Y = \sin\Theta$$

The point (X, Y) then corresponds to the point on the unit circle specified by the angle Θ, as shown in Fig. 7. Previously we saw that the marginal pdf's of X and Y are arcsine pdf's, which are nonzero in the interval $(-1, 1)$. The product of the marginals is nonzero in the square defined by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, so if X and Y were independent the point (X, Y) would assume all values in this square. This is not the case, so X and Y are dependent. We now show that X and Y are uncorrelated:

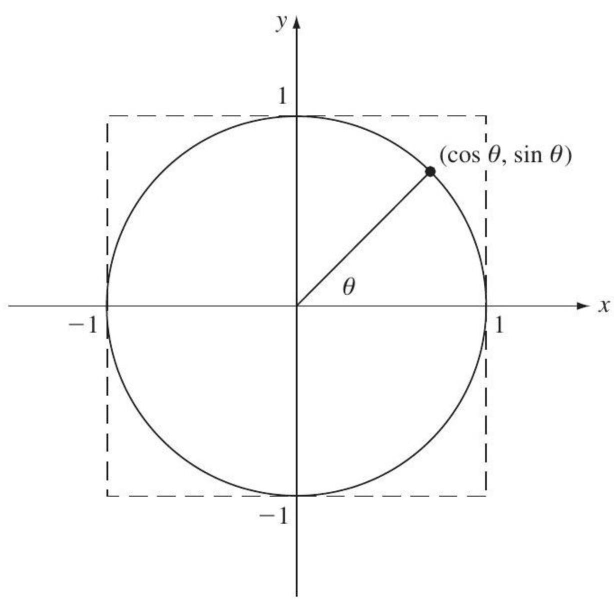

Figure 7: (X, Y) is a point selected at random on the unit circle. X and Y are uncorrelated but not independent.

$$E[XY] = E[\sin\Theta\cos\Theta] = \frac{1}{2\pi}\int_0^{2\pi} \sin\phi\cos\phi d\phi$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \sin 2\phi d\phi = 0$$

Since $E[X] = E[Y] = 0$, Eq. (19) then implies that X and Y are uncorrelated.

Example 14

Let X and Y be the random variables discussed in Example 2. Find $E[XY], \mathrm{COV}(X,Y)$, and $\rho_{X,Y}$.

Equations (19) and (20) require that we find the mean, variance, and correlation of X and Y. From the marginal pdf's of X and Y obtained in Example (6), we find that $E[X] = 3/2$ and $\mathrm{VAR}[X] = 5/4$, and that $E[Y] = 1/2$ and $\mathrm{VAR}[Y] = 1/4$. The correlation of X and Y is

$$\begin{aligned} E[XY] &= \int_0^{\infty} \int_0^{x} xy2e^{-x}e^{-y}dydx \\ &= \int_0^{\infty} 2xe^{-x}\left(1 - e^{-x} - xe^{-x}\right)dx = 1 \end{aligned}$$

Thus the correlation coefficient is given by

$$\rho_{X,Y} = \frac{1 - \frac{3}{2}\frac{1}{2}}{\sqrt{\frac{5}{4}}\sqrt{\frac{1}{4}}} = \frac{1}{\sqrt{5}}$$